Solutions to Math 232 Midterm 1, Version 1 (blue)

Spring 2012, Simon Fraser University

1. (a) $\begin{pmatrix} 2 & 4 & 5 \\ 0 & 0 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 1 \\ 0 & 2 \\ -1 & 3 \end{pmatrix}$

(c) $\frac{1}{6}\begin{pmatrix} 0 & -2 \\ 3 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} -2 & 0 \\ -10 & 0 \end{pmatrix}$

2. (a) $\mathbf{u} \cdot \mathbf{v} = (1, 2, -1) \cdot (3, -2, -1) = (1)(3) + (2)(-2) + (-1)(-1) = 3 - 4 + 1 = 0$. The dot product is zero, so they are orthogonal.
(b) If (x, y, z) is to be orthogonal to $(1, 2, -1)$ and $(3, -2, -1)$, then $(x, y, z) \cdot (1, 2, -1) = 0$ and $(x, y, z) \cdot (3, -2, -1) = 0$. So the system of equations is

$$\begin{aligned} x + 2y - z &= 0 \\ 3x - 2y - z &= 0. \end{aligned}$$

In matrix form, this is

$$\begin{pmatrix} 1 & 2 & -1 \\ 3 & -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(c) Note that $\mathbf{v} \cdot \mathbf{w}_1 = 8$ so $\{\mathbf{u}, \mathbf{v}, \mathbf{w}_1\}$ is not orthogonal.
Likewise, note that $\mathbf{u} \cdot \mathbf{w}_3 = 8$, so $\{\mathbf{u}, \mathbf{v}, \mathbf{w}_3\}$ is not orthogonal.
On the other hand $\mathbf{u} \cdot \mathbf{w}_2 = 0$ and $\mathbf{v} \cdot \mathbf{w}_2 = 0$, and we already showed that $\mathbf{u} \cdot \mathbf{v} = 0$ in part(a), so $\{\mathbf{u}, \mathbf{v}, \mathbf{w}_2\}$ is orthogonal.
(d) We can take our orthogonal set from part (c) and normalize the three vectors to get the orthonormal set $\left\{\frac{1}{||\mathbf{u}||}\mathbf{u}, \frac{1}{||\mathbf{v}||}\mathbf{v}, \frac{1}{||\mathbf{w}_2||}\mathbf{w}_2\right\}$. Calculating

$||\mathbf{u}|| = \sqrt{1^2+2^2+(-1)^2} = \sqrt{6}$, and $||\mathbf{v}|| = \sqrt{3^2+(-2)^2+(-1)^2} = \sqrt{14}$, and $||\mathbf{w}_2|| = \sqrt{2^2+1^2+4^2} = \sqrt{21}$, so our orthonormal set is

$$\left\{ \frac{1}{\sqrt{6}}(1,2,-1), \frac{1}{\sqrt{14}}(3,-2,-1), \frac{1}{\sqrt{21}}(2,1,4) \right\}.$$

3. No explanations were needed, but we give them for study purposes.
(a) True. (See Instructor Problem 2 of Homework 4 for a proof that $A^TA = I$, so A is invertible with inverse A^T.)
(b) False. (See Problem 3.2.D5(d) from the text, on Assignment 4.)
(c) True. (There is at least one free variable.)
(d) True. (By Theorem 3.3.9, comparing (a) with (d) and (f), if A were invertible, then both systems would have exactly one solution.)
(e) False. (See my warning at the end of Lecture Notes 10.)
(f) True. (If the span of S is the same as the span of T, then $\mathbf{v}_k$ must be in the span of T. So $\mathbf{v}_k$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$, and so by Theorem 3.4.6, the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent.)

4. We row reduce the augmented matrix

$$\left(\begin{array}{ccc|c} 2 & -2 & 3a+2 & 4 \\ 1 & a & 1 & 4 \\ 1 & -1 & 4 & a \end{array}\right)$$

We switch the first and third rows

$$\left(\begin{array}{ccc|c} 1 & -1 & 4 & a \\ 1 & a & 1 & 4 \\ 2 & -2 & 3a+2 & 4 \end{array}\right)$$

We subtract the first row from the second

$$\left(\begin{array}{ccc|c} 1 & -1 & 4 & a \\ 0 & a+1 & -3 & 4-a \\ 2 & -2 & 3a+2 & 4 \end{array}\right)$$

We subtract two times the first row from the third

$$\left(\begin{array}{ccc|c} 1 & -1 & 4 & a \\ 0 & a+1 & -3 & 4-a \\ 0 & 0 & 3a-6 & 4-2a \end{array}\right)$$

Now we note that if $a \neq -1$ and $a \neq 2$, the three diagonal entries are all nonzero and so one can see that we will have three pivot variables and no free variables. So there will be precisely one solution when $a \neq -1$ and $a \neq 2$.

If $a = 2$, then we have

$$\left(\begin{array}{ccc|c} 1 & -1 & 4 & 2 \\ 0 & 3 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

and we will have two pivot variables, one free variable, and no inconsistent equations. So the system will have infinitely many solutions when $a = 2$.

If $a = -1$, then we have

$$\left(\begin{array}{ccc|c} 1 & -1 & 4 & -1 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & -9 & 6 \end{array}\right)$$

and then we can subtract three times the second row from the third to obtain

$$\left(\begin{array}{ccc|c} 1 & -1 & 4 & -1 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & -9 \end{array}\right)$$

and the last equation is inconsistent. So the system has no solutions when $a = -1$.

5 (a). We are looking for points (x, y, z) with

$$||(x, y, z) - (-1, 3, -2)|| = ||(x, y, z) - (1, -1, 4)||.$$

Since distances are nonnegative, this condition is the same as

$$||(x, y, z) - (-1, 3, -2)||^2 = ||(x, y, z) - (1, -1, 4)||^2,$$

which, by our distance formula is

$$(x + 1)^2 + (y - 3)^2 + (z + 2)^2 = (x - 1)^2 + (y + 1)^2 + (z - 4)^2$$

and we expand out to get

$$x^2 + 2x + 1 + y^2 - 6y + 9 + z^2 + 4z + 4 = x^2 - 2x + 1 + y^2 + 2y + 1 + z^2 - 8z + 16,$$

and now cancel the x^2 and y^2 and z^2 terms and gather the rest together to get

$$4x - 8y + 12z = 4,$$

or, dividing by 4, we get

$$x - 2y + 3z = 1.$$

(b). We want to find the (x, y, z) satisfying both of

$$\begin{aligned} x - 2y + 3z &= 1 \\ -x + 3y - 2z &= 0, \end{aligned}$$

which in augmented matrix form is

$$\left(\begin{array}{rrr|r} 1 & -2 & 3 & 1 \\ -1 & 3 & -2 & 0 \end{array}\right)$$

so we row reduce. Add the first row to the second to get

$$\left(\begin{array}{rrr|r} 1 & -2 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right)$$

Add two times the second row to the first to get

$$\left(\begin{array}{rrr|r} 1 & 0 & 5 & 3 \\ 0 & 1 & 1 & 1 \end{array}\right)$$

So we can let the free variable z be equal to a parameter t to get the parametric equations

$$\begin{aligned} x &= -5t + 3 \\ y &= -t + 1 \\ z &= t \end{aligned}$$

for $-\infty < t < \infty$. This is equivalent to the vector equation $(x, y, z) = (-5t + 3, -t + 1, t) = (3, 1, 0) + t(-5, -1, 1)$, for $-\infty < t < \infty$.